CERTAIN TRIPLE q-INTEGRAL EQUATIONS INVOLVING THIRD JACKSON q-BESSEL FUNCTIONS AS KERNEL

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Abstract. In this paper, we employ the fractional q-calculus in solving a triple system of q-Integral equations, where the kernel is the third Jackson q-Bessel functions. The solution is reduced to two simultaneous Fredholm q-integral equation of the second kind. Examples are included. We also apply a result in [6] for solutions of dual q^2-integral equations to solve certain triple integral equations.

1. Introduction

Some three-parts mixed boundary value problems of the mathematical theory of elasticity are solved by reducing them to triple integral equations. Many of the triple integral equations are of the form

$$\begin{aligned} \int_0^\infty A(u)K(u,x)\,du &= f(x), \quad 0<x<a,\\ \int_0^\infty w(u)A(u)K(u,x)\,du &= g(x), \quad a<x<b,\\ \int_0^\infty A(u)K(u,x)\,du &= h(x), \quad b<x<\infty, \end{aligned}$$

where $w(u)$ is the weight function, $K(u,x)$ is the kernel function. Several authors have described various methods to solve dual and triple integral equations especially when the kernel is a Bessel function. For the dual integral equations, see for example [13,15,21,24,26,27,29,30]. For the triple integral equations, see for example [9–12,14,25,31,32,34,35]. In this paper, we consider triple q-integral equation where the kernel is the third Jackson q-Bessel function and the q-integral is Jackson q-integral. It is worth mentioning that different approaches for solving dual q-integral equation is in [6]. Also, solutions for dual and triple sequence involving q-orthogonal polynomials is in [7]. This paper is organized as follows. The next section is introductory section includes the main notions and notations we need in our investigations. In Section 3, we solve the triple q-integral equations by reducing the system to two simultaneous Fredholm q-integral equation of the second kind, we shall use a method due to Singh, Rokne and Dhaliwal [31]. The approach depends on fractional q-calculus. We include solutions of two dual q-integral equations as special cases of the solution of the triple q-integral equation included in this section,

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and we show that this coincides with the results in [6]. In the last section, we use a result from [6] for a solution of dual q^2-integral equations to solve triple q^2-integral equations. The result of this section is a q-analogue of the results introduced by cooke in [11].

2. q-Notations and Results

In this paper, we assume that q is a positive number less than one. We introduce some of the needed q-notations and results (see [5]).

Let $t > 0$, $A_{q,t}$, $B_{q,t}$ and $\mathbb{R}_{q,t,+}$ be the sets defined by

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad B_{q,t} := \{tq^{-n} : n \in \mathbb{N}\},$$

$$\mathbb{R}_{q,t,+} := \{tq^k : k \in \mathbb{Z}\},$$

where $\mathbb{N}_0 := \{0, 1, 2, ...\}$, and $\mathbb{N} := \{1, 2, ...\}$. (Note that if $t = 1$, we write A_q, B_q and $\mathbb{R}_{q,+}$). We follow Gasper and Rahman [17] for the definitions of the q-shifted factorial, multiple q-shifted factorials, basic hypergeometric series, Jackson q-integrals, the q-gamma and beta functions. We also follow Annaby and Mansour [5] for the definition of the q-derivative at zero.
Let $\alpha \in \mathbb{C}$, we will use the following notation

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \begin{cases} 1, & k = 0; \\ \frac{(1-q^{\alpha})(1-q^{\alpha-1})\ldots(1-q^{\alpha-k+1})}{(q;q)_k}, & k \in \mathbb{N}. \end{cases}$$

For $\eta \in \mathbb{C}$ and a function f defined on $\mathbb{R}_{q,+}$, we define the following spaces

$$L_{q,\eta}(\mathbb{R}_{q,+}) := \Big\{ f : \|f\|_{q,\eta} := \int_0^{\infty} |t^{\eta} f(t)| d_q t < \infty \Big\},$$

$$L_{q,\eta}(A_q) := \Big\{ f : \|f\|_{A_q,\eta} := \int_0^{1} |t^{\eta} f(t)| d_q t < \infty \Big\},$$

and

$$L_{q,\eta}(B_q) := \Big\{ f : \|f\|_{B_q,\eta} := \int_1^{\infty} |t^{\eta} f(t)| d_q t < \infty \Big\}.$$

Note that $L_{q,\eta}(\mathbb{R}_{q,+}) = L_{q,\eta}(A_q) \cap L_{q,\eta}(B_q)$.
The third Jackson q-Bessel function $J_{\nu}^{(3)}(z;q)$, see [18] and [19], is defined by

$$J_{\nu}(z;q) = J_{\nu}^{(3)}(z;q) : = \frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}} z^{\nu} {}_1\phi_1(0; q^{\nu+1}; q, qz^2) \tag{2.1}$$

$$= \frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2} z^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n}, z \in \mathbb{C}, \tag{2.2}$$

and satisfies the following relations (see [33]):

$$D_q \left[(.)^{-\nu} J_{\nu}(.;q^2) \right](z) = -\frac{q^{1-\nu} z^{-\nu}}{1-q} J_{\nu+1}(qz;q^2), \tag{2.3}$$

$$D_q \left[(.)^{\nu} J_{\nu}(.;q^2) \right](z) = \frac{z^{\nu}}{1-q} J_{\nu-1}(z;q^2). \tag{2.4}$$

Also, for $\Re(\nu) > -1$, the q-Bessel function $J_{\nu}(.;q^2)$ satisfies (see [22]):

$$\left|J_\nu(q^n;q^2)\right| \le \frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases} q^{n\nu}, & \text{if } n\ge 0;\\ q^{n^2-(\nu+1)n}, & \text{if } n<0.\end{cases} \tag{2.5}$$

We recall that the functions $\cos(z;q)$ and $\sin(z;q)$ are defined for $z\in\mathbb{C}$ by

$$\cos(z;q) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\,(zq^{-\frac12}(1-q))^{\frac12}\,J_{-\frac12}(z(1-q)/\sqrt{q};q^2),$$

$$\sin(z;q) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\,(z(1-q))^{\frac12}\,J_{\frac12}(z(1-q);q^2).$$

We need the following results from [5]:

Proposition 2.1. *Let α, $\beta\in\mathbb{C}$, $\rho,t\in\mathbb{R}_{q,+}$. If $\Re(\beta)>\Re(\alpha)>-1$, the*

$$\int_0^\infty t^{\alpha-\beta+1}J_\alpha(\xi t;q^2)J_\beta(\rho t;q^2)\,d_qt$$

$$=\begin{cases} 0, & \xi>\rho;\\ \frac{(1-q)(1-q^2)^{1-\beta+\alpha}}{\Gamma_{q^2}(\beta-\alpha)}\xi^\alpha\rho^{(\beta-2\alpha-2)}(q^2\xi^2/\rho^2;q^2)_{\beta-\alpha-1}, & \xi\le\rho.\end{cases}$$

Proposition 2.2. *Let ν and α be complex numbers such that $\Re(\nu)>-1$. Then for $\rho,u\in\mathbb{R}_{q,+}$*

$$\int_\rho^\infty x^{2\alpha-\nu-1}(\rho^2/x^2;q^2)_{\alpha-1}J_\nu(ux;q^2)\,d_qx = \frac{(1-q)\Gamma_{q^2}(\alpha)}{(1-q^2)^{1-\alpha}}\rho^{\alpha-\nu}u^{-\alpha}q^\alpha J_{\nu-\alpha}(u\rho/q;q^2).$$

Proposition 2.3. *Let x, ν and γ be complex numbers and $u\in\mathbb{R}_{q,+}$. Then, for $\Re(\gamma)>-1$ and $\Re(\nu)>-1$ the following identity holds*

$$\begin{aligned}&\int_0^x \rho^{\nu+1}(q^2\rho^2/x^2;q^2)_\gamma J_\nu(u\rho;q^2)\,d_q\rho =\\ &x^{\nu-\gamma+1}u^{-\gamma-1}(1-q)(1-q^2)^\gamma\Gamma_{q^2}(\gamma+1)J_{\gamma+\nu+1}(ux;q^2).\end{aligned} \tag{2.6}$$

Moreover, if $\Re(\gamma)>0$ and $\Re(\nu)>-1$, then

$$\begin{aligned}&\int_x^\infty \rho^{2\gamma-\nu-1}(x^2/\rho^2;q^2)_{\gamma-1}J_\nu(u\rho;q^2)\,d_q\rho\\ &= x^{\gamma-\nu}u^{-\gamma}(1-q)q^\gamma\frac{(q^2;q^2)_\infty}{(q^{2\gamma};q^2)_\infty}J_{\nu-\gamma}(\frac{ux}{q};q^2).\end{aligned} \tag{2.7}$$

Corollary 2.4. *Let x,u and α be complex numbers such that $u\in\mathbb{R}_{q,+}$, $\Re(\alpha)>-1$ and $\Re(\nu)>-1$. Then*

$$\begin{aligned}&u^\alpha J_{\nu-\alpha}(ux;q^2) =\\ &\frac{(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}x^{\alpha-\nu-1}D_{q,x}\left[x^{-2\alpha}\int_0^x \rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}J_\nu(u\rho;q^2)\,d_q\rho\right].\end{aligned} \tag{2.8}$$

Proof. Applying (2.6) with $\gamma=-\alpha$, we have:

$$\begin{aligned}&\int_0^x \rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}J_\nu(u\rho;q^2)\,d_q\rho\\ &= x^{\nu+\alpha+1}u^{\alpha-1}(1-q)(1-q^2)^{-\alpha}\Gamma_{q^2}(1-\alpha)J_{\nu-\alpha+1}(ux;q^2).\end{aligned} \tag{2.9}$$

Multiply both sides of equation (2.9) by $x^{-2\alpha}$, and then calculate the q-derivative of the two sides with respect to x and using (2.4), we get the required result. □

Corollary 2.5. *Let x, u and α be complex numbers such that $u \in \mathbb{R}_{q,+}$, $\Re(\alpha) > 0$ and $\Re(\nu) > -1$. Then*

$$u^\alpha J_{\nu+\alpha}(ux; q^2) = -\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2} x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)} D_{q,x} \int_x^\infty \rho^{-2\alpha-\nu+1} (x^2/\rho^2; q^2)_{-\alpha} J_\nu(u\rho; q^2)\, d_q\rho. \tag{2.10}$$

Proof. The proof is similar to the proof of Corollary 2.4 and is omitted. □

Koornwinder and Swarttouw [22] introduced the following inverse pair of q-Hankel integral transforms under the side condition $f,\, g \in L_q^2(\mathbb{R}_{q,+})$:

$$g(\lambda) = \int_0^\infty x f(x) J_\nu(\lambda x; q^2)\, d_q x; \quad f(x) = \int_0^\infty \lambda f(\lambda) J_\nu(\lambda x; q^2)\, d_q\lambda, \tag{2.11}$$

where $\lambda,\, x \in \mathbb{R}_{q,+}$.

In the following, we introduce A q-analogue of the Riemann-Liouville fractional integral operator is introduced in [3] by Al-Salam through

$$I_q^\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} f(t)\, d_q t,$$

$\alpha \notin \{-1, -2, ...\}$. In [1], Agarwal defined the q-fractional derivative to be

$$D_q^\alpha f(x) := I_q^{-\alpha} f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x (qt/x; q)_{-\alpha-1} f(t)\, d_q t.$$

We shall also use that

$$I_q^\alpha D_q^\alpha f(x) = f(x) - I_q^{1-\alpha} f(0) \frac{x^{\alpha-1}}{\Gamma_q(\alpha)},\ 0 < \alpha < 1. \tag{2.12}$$

see [5, Lemma 4.17].

In the following, we introduce some q-fractional operators that we use in solving the triple q-integral equations under consideration. The technique of using fractional operators in solving dual and triple integral equations is not new. See for example [2, 6, 30]. In [3], Al-Salam defined a two parameter q-fractional operator by

$$K_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty \big(x/t; q\big)_{\alpha-1} t^{-\eta-1} \phi(tq^{1-\alpha})\, d_q t,$$

$\alpha \neq -1, -2, \ldots$. This is a q-analogue of the Erdélyi and Sneddon fractional operator, cf. [15, 16],

$$K^{\eta,\alpha} f(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty \big(t-x\big)^{\alpha-1} t^{-\eta-1} f(t)\, dt.$$

In [6], the authors introduced a slight modification of the operator $K_q^{\eta,\alpha}$. This operator is denoted by $\mathcal{K}_q^{\eta,\alpha}$ and defined by

$$\mathcal{K}_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (x/t; q)_{\alpha-1} t^{-\eta-1} \phi(qt) d_q t, \tag{2.13}$$

where $\alpha \neq -1, -2, \ldots$. In case of $\eta = -\alpha$, we set

$$\begin{aligned}\mathcal{K}_q^{\alpha} f(x) &:= q^{-\alpha} x^{\alpha} q^{\frac{\alpha(\alpha-1)}{2}} \mathcal{K}_q^{-\alpha,\alpha} f(x) \\ &= \frac{q^{\frac{-\alpha(\alpha-1)}{2}}}{\Gamma_q(\alpha)} \int_x^{\infty} t^{\alpha-1} (x/t;q)_{\alpha-1} f(qt)\, d_q t.\end{aligned} \tag{2.14}$$

This is a slight modification of the operator $K^{\alpha} f(x;q)$ introduced in [18, (19.4.8)] and by Al-salam in [3]. Note that this operator satisfies the following semigroup identity

$$\mathcal{K}_q^{\alpha} \mathcal{K}_q^{\beta} \phi(x) = \mathcal{K}_q^{\alpha+\beta} \phi(x), \qquad \text{for all } \alpha \text{ and } \beta. \tag{2.15}$$

The proof of (2.15) is completely similar to the proof of [5, Theorem 5.13] and is omitted.

Proposition 2.6. *Let $\alpha \in \mathbb{C}, x \in B_q$. If $\Phi \in L_{q,\alpha-1}(B_q)$ and $G(x) = D_{q,x} \mathcal{K}_q^{\alpha} \Phi(x)$, then*

$$\Phi(x) = -q^{\alpha-1} \mathcal{K}_q^{1-\alpha} G(\frac{x}{q}).$$

Proof. First, we show that

$$G(x) = -q^{1-\alpha} \mathcal{K}_q^{(\alpha-1)} \Phi(qx).$$

According to (2.14), we have

$$\begin{aligned} G(x) = \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} D_{q,x} \int_x^{\infty} t^{\alpha-1} (x/t;q)_{\alpha-1} \Phi(qt)\, d_q t \\ = \frac{q^{-\alpha(\alpha-1)/2}}{x(1-q)\Gamma_q(\alpha)} \Big[\int_x^{\infty} t^{\alpha-1} (x/t;q)_{\alpha-1} \Phi(qt)\, d_q t - \int_{qx}^{\infty} t^{\alpha-1} (qx/t;q)_{\alpha-1} \Phi(qt)\, d_q t \Big]. \end{aligned} \tag{2.16}$$

Note that

$$\int_{qx}^{\infty} g(t)\, d_q t = \int_x^{\infty} g(t)\, d_q t + x(1-q) g(x),$$

so, (2.16) can be written as

$$G(x) = \frac{q^{\frac{-\alpha(\alpha-1)}{2}}}{\Gamma_q(\alpha)} \Big[\int_x^{\infty} t^{\alpha-1} \Big(D_{q,x} (x/t;q)_{\alpha-1} \Big) \Phi(qt)\, d_q t - x^{\alpha-1} (q;q)_{\alpha-1} \Phi(qx) \Big].$$

But

$$D_{q,x} (x/t;q)_{\alpha-1} = -\frac{(1-q^{\alpha-1})}{t(1-q)} (qx/t;q)_{\alpha-2} = -\frac{1}{t} [\alpha-1] (qx/t;q)_{\alpha-2}.$$

Hence,

$$\begin{aligned} G(x) &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} [\alpha-1] \int_x^{\infty} t^{\alpha-2} (qx/t;q)_{\alpha-2} \Phi(qt)\, d_q t - x^{\alpha-1} (q;q)_{\alpha-1} \Phi(qx) \\ &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} [\alpha-1] \int_{qx}^{\infty} t^{\alpha-2} (qx/t;q)_{\alpha-2} \Phi(qt)\, d_q t \\ &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha-1)} \int_{qx}^{\infty} t^{\alpha-2} (qx/t;q)_{\alpha-2} \Phi(qt)\, d_q t = -q^{1-\alpha} \mathcal{K}_q^{(\alpha-1)} \Phi(qx) \end{aligned}$$

This implies,

$$\mathcal{K}_q^{(\alpha-1)} \Phi(x) = -q^{\alpha-1} G(x/q),$$

and by using (2.15), we obtain the result and completes the proof. $\square$

3. **A system of triple q-Integral Equations**

The goal of this section is to solve the following triple q-integral equations:

$$\int_0^\infty \psi(u) J_\nu(u\rho;q^2)\,d_qu = f_1(\rho), \quad \rho \in A_{q,a}, \tag{3.1}$$

$$\int_0^\infty u^{-2\alpha}\psi(u)\left[1+w(u)\right] J_\nu(u\rho;q^2)\,d_qu = f_2(\rho), \quad \rho \in A_{q,b}\cap B_{q,a}, \tag{3.2}$$

$$\int_0^\infty \psi(u) J_\nu(u\rho;q^2)\,d_qu = f_3(\rho), \quad \rho \in B_{q,b}, \tag{3.3}$$

where $0<a<b<\infty$, and α, ν are complex numbers satisfying

$$\Re(\nu) > -1, \quad \text{and} \quad 0<\Re(\alpha)<1.$$

ψ is an unknown function to be determined, and f_i $(i=1,2,3)$ are known functions, and w is a non-negative bounded function defined on $\mathbb{R}_{q,+}$.

Clearly from (2.5), a sufficient condition for the convergence of the q-integrals on the left hand side of (3.1)–(3.2) is that

$$\psi \in L_{q,\nu}(\mathbb{R}_{q,+}) \cap L_{q,\nu-2\alpha}(\mathbb{R}_{q,+}). \tag{3.4}$$

For getting the solution of the triple q-integral equations (3.1)–(3.3), we define a function C by

$$C(u) := u^{-2\alpha}\psi(u)\left[1+w(u)\right], \quad u \in \mathbb{R}_{q,+}.$$

Hence,

$$\psi(u) = u^{2\alpha}C(u) - u^{2\alpha}C(u)\left[\frac{w(u)}{1+w(u)}\right],$$

and the triple q-integral equation (3.1)–(3.3) can be represented as:

$$\begin{aligned}\int_0^\infty u^{2\alpha}C(u)J_\nu(u\rho;q^2)\,d_qu - \int_0^\infty u^{2\alpha}C(u)\left[\frac{w(u)}{1+w(u)}\right]J_\nu(u\rho;q^2)\,d_qu \\ = f_1(\rho), \qquad \rho\in A_{q,a}\end{aligned} \tag{3.5}$$

$$\int_0^\infty C(u)J_\nu(u\rho;q^2)\,d_qu = f_2(\rho), \quad \rho \in A_{q,b}\cap B_{q,a} \tag{3.6}$$

$$\begin{aligned}\int_0^\infty u^{2\alpha}C(u)J_\nu(u\rho;q^2)\,d_qu - \int_0^\infty \frac{w(u)}{1+w(u)}u^{2\alpha}C(u)J_\nu(u\rho;q^2)\,d_qu \\ = f_3(\rho), \quad \rho\in B_{q,b}\end{aligned} \tag{3.7}$$

Now assume that $C := C_1 + C_2$ such that

$$\int_0^\infty C_1(u)J_\nu(u\rho;q^2)\,d_qu = g_1(\rho),\ \rho\in A_{q,b},$$

$$\int_0^\infty C_2(u)J_\nu(u\rho;q^2)\,d_qu = g_2(\rho),\ \rho\in B_{q,a},$$

where

$$f_1(\rho) = g_1(\rho) + g_2(\rho), \quad \rho\in A_{q,b}\cap B_{q,a}.$$

So, the triple q-integral equations (3.5)–(3.7) can be rewritten in the following form:

$$\begin{aligned}\int_0^\infty u^{2\alpha}\left[C_1(u)+C_2(u)\right]J_\nu(u\rho;q^2)\,d_qu-\\ \int_0^\infty u^{2\alpha}\left[C_1(u)+C_2(u)\right]\frac{w(u)}{1+w(u)}J_\nu(u\rho;q^2)\,d_qu=f_1(\rho),\quad \rho\in A_{q,a},\end{aligned}\tag{3.8}$$

$$\int_0^\infty C_1(u)J_\nu(u\rho;q^2)\,d_qu=g_1(\rho),\quad \rho\in A_{q,b},\tag{3.9}$$

$$\int_0^\infty C_2(u)J_\nu(u\rho;q^2)\,d_qu=g_2(\rho),\quad \rho\in B_{q,a},\tag{3.10}$$

$$\begin{aligned}\int_0^\infty u^{2\alpha}\left[C_1(u)+C_2(u)\right]J_\nu(u\rho;q^2)\,d_qu-\\ \int_0^\infty \frac{w(u)}{1+w(u)}u^{2\alpha}\left[C_1(u)+C_2(u)\right]J_\nu(u\rho;q^2)\,d_qu=f_3(\rho),\quad \rho\in B_{q,b}.\end{aligned}\tag{3.11}$$

Proposition 3.1. *Let the functions ψ_1, ψ_2 be defined by*

$$\psi_1(x):=\int_0^\infty u^\alpha C_1(u)J_{\nu-\alpha}(ux;q^2)\,d_qu,\quad x\in B_{q,b},\tag{3.12}$$

$$\psi_2(x):=\int_0^\infty u^\alpha C_2(u)J_{\nu+\alpha}(ux;q^2)\,d_qu,\quad x\in A_{q,a},\tag{3.13}$$

provided that $0<\Re\alpha<1$, $\Re\nu>-1$, $\Re(\nu+\alpha)>0$ and $C_1\in L_{q,\nu}(\mathbb{R}_{q,+})$, $C_2\in L_{q,-t}(\mathbb{R}_{q,+})$ where

$$\Re\nu+2>\Re t>-\Re\nu+2\Re(1-\alpha).$$

Then for $u\in\mathbb{R}_{q,+}$,

$$C_1(u)=u^{1-\alpha}\left[\int_0^b x\Phi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx+\int_b^\infty x\Psi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx\right],\tag{3.14}$$

$$C_2(u)=u^{1-\alpha}\left[\int_0^a x\Psi_2(x)J_{\nu+\alpha}(ux;q^2)\,d_qx+\int_a^\infty x\Phi_2(x)J_{\nu+\alpha}(ux;q^2)\,d_qx\right],\tag{3.15}$$

where

$$\begin{aligned}\Phi_1(x)&=\frac{(1-q^2)^\alpha x^{\alpha-\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\left[x^{-2\alpha}\int_0^x g_1(\rho)\rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}\,d_q\rho\right]\\ &=(1-q^2)^\alpha x^{\alpha-\nu-1}D^\alpha_{q^2,x}\left(t^{\nu/2}g_1(\sqrt{t})\right)(x),\ x\in A_{q,b},\end{aligned}\tag{3.16}$$

$$\begin{aligned}\Phi_2(x)&=-\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2}x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\int_x^\infty g_2(\rho)\rho^{1-2\alpha-\nu}(x^2/\rho^2;q^2)_{-\alpha}\,d_q\rho,\\ &=-q^{\frac{\alpha(1-\alpha)}{2}}(1-q^2)^\alpha x^{\alpha+\nu-1}D_{q^2,x}\mathcal{K}^{(1-\alpha)}_{q^2}\left[t^{-\nu/2}g_2(\sqrt{t})\right](\frac{x}{q^2}),\ x\in B_{q,a}.\end{aligned}\tag{3.17}$$

Proof. We start with proving (3.16). Let $x \in A_{q,b}$. Multiply both sides of (3.9) by $x^{-2\alpha}\rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}$ and integrate with respect to ρ from 0 to x, we get

$$\begin{aligned}\int_0^x x^{-2\alpha}\rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}\int_0^\infty C_1(u)J_\nu(u\rho;q^2)\,d_qu\,d_q\rho =\\ \int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}\,d_q\rho.\end{aligned} \tag{3.18}$$

We can prove that the double q-integral on the left hand side of (3.18) is absolutely convergent for $0 < \Re(\alpha) < 1$ and for $\Re(\nu) > -1$ provided that $C_1 \in L_{q,\nu}(\mathbb{R}_{q,+})$. So, we can interchange the order of the q-integrations to obtain

$$\begin{aligned}\int_0^\infty C_1(u)x^{-2\alpha}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};q^2)_{-\alpha}J_\nu(u\rho;q^2)\,d_q\rho\,d_qu =\\ \int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}(\frac{q^2\rho^2}{x^2};q^2)_{-\alpha}\,d_q\rho.\end{aligned} \tag{3.19}$$

Calculate the q-derivative of the two sides of (3.19) with respect to x and using (2.8), we get

$$\int_0^\infty u^\alpha C_1(u)J_{\nu-\alpha}(ux;q^2)\,d_qu = \Phi_1(x), \quad x \in A_{q,b}, \tag{3.20}$$

where

$$\Phi_1(x) = \frac{(1-q^2)^\alpha x^{\alpha-\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\left[x^{-2\alpha}\int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}(\frac{q^2\rho^2}{x^2};q^2)_{-\alpha}\,d_q\rho\right].$$

Now, we prove (3.17). Let $x \in B_{q,a}$. Multiply both sides of (3.10) by $\rho^{-2\alpha-\nu+1}(x^2/\rho^2;q^2)_{-\alpha}$ and q-integrate with respect to ρ from x to ∞, we get

$$\begin{aligned}\int_x^\infty \rho^{-2\alpha-\nu+1}(x^2/\rho^2;q^2)_{-\alpha}\int_0^\infty C_2(u)J_\nu(u\rho;q^2)\,d_qu\,d_q\rho =\\ \int_x^\infty g_2(\rho)\rho^{-2\alpha-\nu+1}(x^2/\rho^2;q^2)_{-\alpha}\,d_q\rho.\end{aligned} \tag{3.21}$$

From (2.5), we can prove that $u^t J_\nu(u;q^2)$ is bounded on $R_{q,+}$ provided that $\Re(t+\nu) > -1$. So, if we take t such that $\Re\nu + 2 > \Re t > -\Re\nu + 2\Re(1-\alpha)$, we can prove that the double q-integral

$$\int_x^\infty \rho^{1-2\alpha-\nu}(x^2/\rho^2;q^2)_{-\alpha}\int_0^\infty C_2(u)J_\nu(u\rho;q^2)\,d_qu\,d_q\rho$$

is absolutely convergent and we can interchange the order of the q-integration to obtain

$$\begin{aligned}\int_0^\infty C_2(u)\int_x^\infty \rho^{1-2\alpha-\nu}(x^2/\rho^2;q^2)_{-\alpha}J_\nu(u\rho;q^2)\,d_q\rho\,d_qu\\ = \int_x^\infty g_2(\rho)\rho^{-2\alpha-\nu+1}(x^2/\rho^2;q^2)_{-\alpha}\,d_q\rho.\end{aligned} \tag{3.22}$$

Calculating the q-derivative of the two sides of (3.22) with respect to x and using (2.10) yields

$$\int_0^\infty u^\alpha C_2(u)J_{\nu+\alpha}(ux;q^2)\,d_qu = \Phi_2(x), \quad x \in B_{q,a}, \tag{3.23}$$

where

$$\Phi_2(x) = -\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2} x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)} D_{q,x} \int_x^\infty g_2(\rho)\rho^{1-2\alpha-\nu}(x^2/\rho^2;q^2)_{-\alpha}\, d_q\rho.$$

By the above argument, If we assume that ψ_1 and ψ_2 are given by (3.12) and (3.13), then

$$\int_0^\infty u^\alpha C_1(u) J_{\nu-\alpha}(ux;q^2)\, d_q x = \begin{cases} \phi_1(x), & x \in A_{q,b}, \\ \psi_1(x), & x \in B_{q,b}, \end{cases} \tag{3.24}$$

and

$$\int_0^\infty u^\alpha C_2(u) J_{\nu+\alpha}(ux;q^2)\, d_q x = \begin{cases} \phi_2(x), & x \in B_{q,a}, \\ \psi_2(x), & x \in A_{q,a}. \end{cases} \tag{3.25}$$

Hence, (3.14) and (3.15) follow by applying the inverse pair of q-Hankel transforms (2.11) on (3.24) and (3.25). This completes the proof. □

Remark 3.2. From the definitions of ψ_i and ϕ_i, $i = 1, 2$, in Proposition 3.1, one can verify that $x^{-\nu-\alpha}\phi_2$ is bounded function in $B_{q,a}$ and and $x^{-\nu-\alpha}\psi_2$ is bounded in $A_{q,a}$. Also, $x^{-\nu+\alpha}\phi_1$ is bounded in $A_{q,b}$ and $x^{-\nu+\alpha}\psi_1$ is bounded in $B_{q,b}$.

Proposition 3.3. *For $\rho \in B_{q,b}$, $\Psi_1(\rho)$ satisfies the Fredholm q-integral equation of the form*

$$\psi_1(\rho) = \tilde{F}_1(\rho) + \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2} \int_b^\infty x\psi_1(x) K_1(\rho, x)\, d_q x, \tag{3.26}$$

where

$$K_1(\rho, x) = \int_0^\infty \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u,$$

$$\tilde{F}_1(\rho) = F_1(\rho) -$$

$$\frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2} \int_0^a x\,\psi_2(x) \int_0^\infty \frac{u}{1+w(u)} J_{\nu+\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x,$$

and

$$F_1(\rho) = \rho^{\nu-\alpha} \frac{q^{-2\alpha^2-\alpha+\nu}(1+q)(1-q^2)^{-\alpha}}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_\rho^\infty x^{2\alpha-\nu-1} f_3(qx)(\rho^2/x^2;q^2)_{\alpha-1}\, d_q x -$$

$$\frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2} \Big[\int_a^\infty x\Phi_2(x) \int_0^\infty \frac{u}{1+w(u)} J_{\nu+\alpha}(ux;q^2)\; J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x$$

$$+ \int_0^b x\Phi_1(x) \int_0^\infty \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x \Big].$$

Proof. Equation (3.11) can be written in the following form:

$$\int_0^\infty u^{2\alpha} C_1(u) J_\nu(u\rho;q^2)\, d_q u = G(\rho), \quad \rho \in B_{q,b}, \tag{3.27}$$

where

$$\begin{aligned} G(\rho) = f_3(\rho) &- \int_0^\infty u^{2\alpha} C_2(u) \frac{1}{1+w(u)} J_\nu(u\rho;q^2)\, d_q u \\ &+ \int_0^\infty u^{2\alpha} C_1(u) \frac{w(u)}{1+w(u)} J_\nu(u\rho;q^2)\, d_q u. \end{aligned} \tag{3.28}$$

By using equations (2.3) and (3.27), we get

$$G(\rho) = -(1-q)\rho^{\nu-1}q^{\nu-1}D_{q,\rho}\,\rho^{1-\nu}\int_0^\infty u^{2\alpha-1}C_1(u)J_{\nu-1}(u\rho q^{-1};q^2)\,d_qu. \tag{3.29}$$

Substituting the value of $C_1(u)$ from (3.14) into (3.29), we obtain
(3.30)

$$\begin{gathered}D_{q,\rho}\,\rho^{1-\nu}\int_0^\infty u^\alpha\left[\int_0^b x\Phi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx + \int_b^\infty x\Psi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx\right]\times\\ J_{\nu-1}(u\rho q^{-1};q^2)\,d_qu = -\frac{\rho^{1-\nu}q^{1-\nu}G(\rho)}{(1-q)},\quad \rho\in B_{q,b}.\end{gathered}$$

From (2.5), there exists $M>0$ such that

$$\left|J_{\nu-\alpha}(ux;q^2)\right| \le M(ux)^{\Re(\nu-\alpha)}\ \text{ for all } u,x\in\mathbb{R}_{q^2,b,+},$$

From Remark 3.2, we have

$$\begin{aligned}&\left|\int_0^\infty u^\alpha\left[\int_0^b x\Phi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx + \int_b^\infty x\Psi_1(x)J_{\nu-\alpha}(ux;q^2)\,d_qx\right]J_{\nu-1}(u\rho q^{-1};q^2)\,d_qu\right|\\ &\qquad\le M\left[\left\|\Psi_1(x)\right\|_{A_{q,b,\nu-\alpha}} + \left\|\Phi_1(x)\right\|_{B_{q,b,\nu-\alpha}}\right]\left|\int_0^\infty u^{2\alpha+\nu}J_{\nu+1}(u\rho;q^2)\,d_qu\right| < \infty.\end{aligned}$$

Hence, the double q-integration is absolutely convergent and we can interchange the order of the q-integrations to obtain

$$\begin{aligned}G(\rho) &= -(1-q)\rho^{\nu-1}q^{\nu-1}\left[\int_0^b x\Phi_1(x)\,d_qx + \int_b^\infty x\Psi_1(x)\,d_qx\right]\times\\ &\quad D_{q,\rho}\,\rho^{1-\nu}\int_0^\infty u^\alpha J_{\nu-1}(u\rho q^{-1};q^2)J_{\nu-\alpha}(ux;q^2)\,d_qu,\ \rho\in B_{q,b}.\end{aligned} \tag{3.31}$$

Therefore, applying Proposition 2.1 with $\Re(\nu-\alpha)>\Re(\nu-1)>-1$, we obtain

$$G(\rho) = \frac{-(1-q)^2(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}\rho^{\nu-1}D_{q,\rho}\int_\rho^\infty x^{1-\nu-\alpha}\Psi_1(x)(\rho^2/x^2;q^2)_{-\alpha}\,d_qx. \tag{3.32}$$

By using

$$\int_x^\infty f(t)\,d_qt = \frac{1}{1+q}\int_{x^2}^\infty \frac{f(\sqrt{t})}{\sqrt{t}}\,d_{q^2}t,\quad D_{q,\rho}(f(\rho^2)) = \rho(1+q)\left(D_{q^2}f\right)(\rho^2), \tag{3.33}$$

we obtain

$$\begin{aligned}G(\rho) &= \frac{-(1-q)^2(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}\rho^\nu D_{q^2,\rho^2}\int_{\rho^2}^\infty x^{\frac{-(\nu+\alpha)}{2}}\Psi_1(\sqrt{x})(\rho^2/x;q^2)_{-\alpha}\,d_{q^2}x\\ &= -(1-q)^2(1-q^2)^\alpha q^{\alpha^2-2\alpha-\nu}\rho^\nu\left(D_{q^2}\mathcal{K}_{q^2}^{1-\alpha}\left((\cdot)^{-\frac{\nu+\alpha}{2}}\psi_1(\cdot)\right)\right)(\rho^2/q^2).\end{aligned}$$

Replacing ρ by $q\rho$ yields

$$-q^{-\alpha^2+\alpha}(1-q^2)^{-\alpha}(1-q)^{-2}\left[(\cdot)^{-\nu/2}G(q\sqrt{\cdot})\right](\rho^2) = D_{q^2,\rho^2}\mathcal{K}_{q^2}^{1-\alpha}\left[(\sqrt{\cdot})^{(\alpha-\nu)}\psi_1(\sqrt{\cdot})\right](\rho^2).$$

Thus, applying Proposition 3.3 yields

$$\begin{aligned}\rho^{\alpha-\nu}\Psi_1(\rho) &= q^{-\alpha^2}(1-q)^{-2}(1-q^2)^{-\alpha}\mathcal{K}_{q^2}^{\alpha}\left[(\cdot)^{-\nu/2}G(q\sqrt{\cdot})\right](\rho^2/q^2)\\ &= \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1-q)^{-2}}{\Gamma_{q^2}(\alpha)}\int_{\rho^2}^\infty x^{-\frac{\nu}{2}+\alpha-1}G(q\sqrt{x})(\rho^2/x;q^2)_{\alpha-1}\,d_{q^2}x.\end{aligned}$$

Using $\int_{x^2}^{\infty} f(t)\, d_{q^2}t = (1+q)\int_x^{\infty} t f(t^2)\, d_q t$, we obtain

$$\rho^{\alpha-\nu}\Psi_1(\rho) = \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_{\rho}^{\infty} x^{2\alpha-\nu-1} G(qx)(\rho^2/x^2;q^2)_{\alpha-1}\, d_q x.$$

From (3.28), we can write the last equation in the following form

(3.34)

$$\Psi_1(\rho) + \rho^{\nu-\alpha}\, \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_{\rho}^{\infty} x^{2\alpha-\nu-1}(\rho^2/x^2;q^2)_{\alpha-1}\times$$
$$\Big[\int_0^{\infty} \frac{u^{2\alpha}}{1+w(u)} C_2(u) J_\nu(qux;q^2)\, d_q u - \int_0^{\infty} \frac{w(u)}{1+w(u)} u^{2\alpha} C_1(u) J_\nu(qux;q^2)\, d_q u\Big]\, d_q x =$$
$$\rho^{\nu-\alpha}\, \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_{\rho}^{\infty} x^{2\alpha-\nu-1}\, f_3(qx)(\rho^2/x^2;q^2)_{\alpha-1}\, d_q x,\ \ \rho\in B_{q,b}.$$

From the condition on the function C_2, we can prove that the double q-integration

$$\int_{\rho}^{\infty} x^{2\alpha-\nu-1}(\rho^2/x^2;q^2)_{\alpha-1} \int_0^{\infty} C_2(u) \frac{u^{2\alpha}}{1+w(u)} J_\nu(qux;q^2)\, d_q u\, d_q x$$

is absolutely convergent. Therefore, we can interchange the order of the q-integrations and use Proposition 2.2 to obtain

(3.35)

$$\Psi_1(\rho) + \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2}\Big[\int_0^{\infty} \frac{u^{\alpha}}{1+w(u)} C_2(u) J_{\nu-\alpha}(u\rho;q^2)\, d_q u -$$
$$\int_0^{\infty} \frac{u^{\alpha} w(u)}{1+w(u)} C_1(u) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\Big] =$$
$$\rho^{\nu-\alpha}\, \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_{\rho}^{\infty} x^{2\alpha-\nu-1}\, f_3(qx)(\rho^2/x^2;q^2)_{\alpha-1}\, d_q x,\ \ \rho\in B_{q,b}.$$

Substitute the value of $C_1(u)$ and $C_2(u)$ from equations (3.15) and (3.14) into equation (3.35), and then interchange the order of the q-integrations we get

(3.36)

$$\Psi_1(\rho) + \frac{q^{\nu-4\alpha}}{(1-q)^2}\Big[\int_0^{a} x\psi_2(x) \int_0^{\infty} \frac{u}{1+w(u)} J_{\nu+\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x$$
$$- \int_b^{\infty} x\psi_1(x) \int_0^{\infty} \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x\Big] = F_1(\rho),\ \ \rho\in B_{q,b}.$$

where

$$F_1(\rho) = \rho^{\nu+\alpha}\, \frac{q^{\nu-4\alpha}(1+q)(1-q^2)^{-\alpha}}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_{\rho}^{\infty} x^{2\alpha-\nu-1} f_3(qx)(\rho^2/x^2;q^2)_{\alpha-1}\, d_q x -$$
$$\frac{q^{\nu-4\alpha}}{(1-q)^2}\Big[\int_a^{\infty} x\Phi_2(x) \int_0^{\infty} \frac{u}{1+w(u)} J_{\nu+\alpha}(ux;q^2)\ J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x$$
$$+ \int_0^{b} x\Phi_1(x) \int_0^{\infty} \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}(ux;q^2) J_{\nu-\alpha}(u\rho;q^2)\, d_q u\, d_q x\Big].$$

Equation (3.36) is nothing else but the Fredholm q-integral equation of the second kind (3.26). This completes the proof. □

Proposition 3.4. *For $\rho \in A_{q,a}$, $\Psi_2(\rho)$ satisfies the Fredholm q-integral equation of the form*

$$\psi_2(\rho) = \tilde{F}_2(\rho) + \frac{1}{(1-q)^2} \int_0^a x K_2(\rho, x) \psi_2(x)\, d_q x, \tag{3.37}$$

where

$$K_2(\rho, x) = \int_0^\infty \frac{u w(u)}{1 + w(u)} J_{\nu+\alpha}(ux; q^2) J_{\nu+\alpha}(u\rho; q^2)\, d_q u,$$

$$\tilde{F}_2(\rho) = F_2(\rho) -$$

$$\frac{1}{(1-q)^2} \int_b^\infty x \Psi_1(x) \int_0^\infty \frac{u}{1 + w(u)} J_{\nu-\alpha}(ux; q^2) J_{\nu+\alpha}(u\rho; q^2)\, d_q u\, d_q x$$

and

$$F_2(\rho) = \frac{(1-q^2)^{-\alpha}(1+q)\rho^{\alpha-\nu-2}}{(1-q)^2 \Gamma_{q^2}(\alpha)} \int_0^\rho (q^2 x^2/\rho^2; q^2)_{\alpha-1} x^{\nu+1} f_1(x)\, d_q x +$$

$$\frac{1}{(1-q)^2} \int_a^\infty x \Phi_2(x) \int_0^\infty \frac{u w(u)}{1 + w(u)} J_{\nu+\alpha}(ux; q^2) J_{\nu+\alpha}(u\rho; q^2)\, d_q u\, d_q x -$$

$$\frac{1}{(1-q)^2} \int_0^b x \Phi_1(x) \int_0^\infty \frac{u}{1 + w(u)} J_{\nu-\alpha}(ux; q^2) J_{\nu+\alpha}(u\rho; q^2)\, d_q u\, d_q x.$$

Proof. The proof is similar to the proof of Proposition 3.3 and is omitted. □

Theorem 3.5. *The solution of* (3.1)–(3.2) *is given by*

$$\Psi(u) = \frac{u^{2\alpha}}{1 + w(u)} \left(C_1(u) + C_2(u)\right).$$

The functions C_1, C_2, ϕ_1 and ϕ_2 are given by Proposition 3.1, and ψ_1, ψ_2 satisfies the Fredholm q-integral equations (3.37) *and* (3.26) *of second kind.*

Example 1

1. Take $b = aq^{-m}$ and assume that $m \to \infty$. If we assume that $f_1 = f$, $f_2 = f$, and $w = 0$. Then the system (3.1)–(3.3) is reduced to the dual q-integral equations

$$\int_0^\infty \psi(u) J_\nu(u\rho; q^2)\, d_q u = f(\rho), \quad \rho \in A_{q,a} \tag{3.38}$$

$$\int_0^\infty u^{-2\alpha} \psi(u) J_\nu(u\rho; q^2)\, d_q u = 0, \quad \rho \in B_{q,a}. \tag{3.39}$$

Hence, from Theorem 3.5

$$\psi(u) = u^{1+\alpha} \int_0^\infty x \psi_2(x) J_{\nu+\alpha}(ux; q^2)\, d_q x,\ u \in \mathbb{R}_{q,+}$$

$$\begin{aligned}\psi_2(\rho) &= \frac{(1-q^2)^{-\alpha}(1+q)\rho^{\alpha-\nu-2}}{(1-q)^2 \Gamma_{q^2}(\alpha)} \int_0^\rho (q^2 x^2/\rho^2; q^2)_{\alpha-1} x^{\nu+1} f(x)\, d_q x \\ &= \rho^{-\alpha-\nu} \frac{(1-q^2)^{-\alpha}}{(1-q)^2} I_{q^2}^\alpha \left(t^{\nu/2} f(\sqrt{t})\right)(\rho^2).\end{aligned}$$

Hence,

$$\psi(u) = u^{1+\alpha}\frac{(1-q^2)^{-\alpha}}{(1-q^2)}\int_0^\infty x^{1-\alpha-\nu} I_{q^2}^{\alpha}\left(t^{\nu/2} f(\sqrt{t})\right)(x^2) J_{\nu+\alpha}(ux;q^2)\, d_q x.$$

This coincides with the result in [6, Theorem 4.1] for solutions of double q-integral equations.

Let $a = q^m$ and assume that $m \to \infty$. If we assume that $f_2 = 0$, and $f_3 = f$, we obtain the dual q-integral system of equations

$$\int_0^\infty u^{-2\alpha}\psi(u) J_\nu(u\rho;q^2)\, d_q u = 0, \quad \rho \in A_{q,b} \tag{3.40}$$

$$\int_0^\infty \psi(u) J_\nu(u\rho;q^2)\, d_q u = f, \quad \rho \in B_{q,b}. \tag{3.41}$$

Hence, from Theorem 3.5

$$\psi(u) = u^{1+\alpha}\int_b^\infty x\psi_1(x) J_{\nu-\alpha}(ux;q^2)\, d_q x,\ u \in \mathbb{R}_{q,+},$$

$$\psi_1(\rho) = -\frac{(1-q^2)^{-\alpha} q^{-2\alpha}\rho^{\alpha+\nu}}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_\rho^\infty (\rho^2/x^2;q^2)_{\alpha-1} x^{2\alpha-\nu-1} f(x)\, d_q x.$$

This is a special case of [6, Theorem 5.1].

Example 2

We consider the triple q-integral equations

$$\int_0^\infty \psi(u) J_0(u\rho;q^2)\, d_q u = 0, \quad \rho \in A_{q,a}, \tag{3.42}$$

$$\int_0^\infty u^{-1}\psi(u) J_0(u\rho;q^2)\, d_q u = 1, \quad \rho \in A_{q,b}\cap B_{q,a}, \tag{3.43}$$

$$\int_0^\infty \psi(u) J_0(u\rho;q^2)\, d_q u = 0, \quad \rho \in B_{q,b}. \tag{3.44}$$

Hence, we have $\nu = 0$, $g_1 = 1$, $g_2 = 0$, $f_1 = f_3 = 0$, $w = 0$, and $\alpha = \frac{1}{2}$.

From Theorem 3.5,

$$\psi(u) = u\left(C_1(u) + C_2(u)\right),$$

where

$$C_1(u) = \frac{(1-q)(1-q^2)}{\Gamma_{q^2}^2(1/2)}\frac{\sin(\frac{bu}{1-q};q)}{u} + \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)}\int_b^\infty \sqrt{x}\psi_1(x)\cos(\frac{xu\sqrt{q}}{1-q};q^2)\, d_q x,$$

$$C_2(u) = \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)}\int_0^a \sqrt{x}\psi_2(x)\sin(\frac{xu}{1-q};q^2)\, d_q x,$$

$$\psi_1(\rho) = \frac{\sqrt{\rho}(1+q)}{q(1-q)\Gamma_{q^2}^2(1/2)}\int_0^a x^{3/2}\frac{\psi_2(x)}{q\rho^2 - x^2}\, d_q x,\ \rho \in B_{q,b}, \tag{3.45}$$

$$\begin{aligned}\psi_2(\rho) = &-\frac{(1+q)\sqrt{\rho}}{(1-q)\Gamma_{q^2}^2(1/2)}\int_b^\infty \frac{\sqrt{x}\psi_1(x)}{qx^2-\rho^2}\, d_q x\\ &+\frac{(1+q)^{3/2}}{\sqrt{1-q}\,\Gamma_{q^2}^3(1/2)}\sqrt{\rho}\int_{\rho/q}^b \frac{d_q x}{qx^2-\rho^2}.\end{aligned} \tag{3.46}$$

We used [22, PP. 455-466] or [6, Proposition 2.4] to calculate ψ_1 and ψ_2 in equations (3.45) and (3.46), respectively. Substituting from (??) into (??), we obtain the second order Fredholm q-integral equation

(3.47)
$$\psi_2(\rho) = -\frac{q^{-1}\sqrt{\rho}(1+q)}{(1-q)^2\Gamma^3_{q^2}(1/2)}\int_0^a t^{3/2}\psi_2(t)K_2(\rho,t)\,d_qt + \frac{(1+q)^{3/2}}{\sqrt{1-q}\Gamma^3_{q^2}(1/2)}\sqrt{\rho}\int_{\rho/q}^b \frac{d_qx}{qx^2-\rho^2}.,$$

where $\rho \in A_{q,a}$ and

$$K(\rho,t) = \int_b^\infty \frac{x}{(t^2-qx^2)(\rho^2-qx^2)}\,d_qt.$$

4. Solving system of triple q^2-Integral Equations by using solutions of dual q-integral equations

In [11], Cooke solved certain triple integral equations involving Bessel functions by using a result for Noble [28] for solutions for dual integral equations with Bessel functions as kernel. In this section, we use the result, Theorem A , which introduced in [6] to solve the following triple q-integral equations:

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}\psi(\rho)J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f(\xi), \quad \xi \in A_{q^2} \tag{4.1}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g(\xi), \quad \xi \in A_{q^2}\cap B_{q^2} \tag{4.2}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}\psi(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = h(\xi), \quad \xi \in B_{q^2} \tag{4.3}$$

where a, α, β, γ, μ, ν and κ are complex numbers such that

$$\Re(\nu) > -1,\ \ \Re(\mu) > -1,\ \ \Re(\kappa) > -1, \text{ and } 0 < a < 1,$$

, the functions $f(\rho)$, $g(\rho)$ and $h(\rho)$ are known functions, and $\psi(u)$ is the solution function to be determined.

The following is a result from [6] that we shall use to solve the system (4.1)–(4.3).

Theorem A. *Let α, β, μ and ν be complex numbers and let $\lambda := \frac{1}{2}(\mu+\nu)-(\alpha-\beta) > -1$. Assume that*

$$\Re(\nu) > -1, \Re(\mu) > -1, \Re(\lambda) > -1, \quad \text{and } \Re(\lambda-\mu-2\alpha) > 0.$$

Let $f \in L_{q^2,\frac{\mu}{2}+\alpha}(A_{q^2})$ and $g \in L_{q^2,\frac{-\mu}{2}+\alpha-1}(B_{q^2})$. Then the dual q^2-integral equations

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f(\xi), \quad \xi \in A_{q^2}, \tag{4.4}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}\psi(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g(\xi), \quad \xi \in B_{q^2} \tag{4.5}$$

has the solution of the form

$$\psi(\xi) = (1-q^2)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda(\sqrt{\rho\xi};q^2)I_{q^2}^{\mu/2+\alpha,\lambda-\mu}f(\rho)\,d_{q^2}\rho$$
$$+(1-q^2)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda(\sqrt{\rho\xi};q^2)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda}g(\rho)\,d_{q^2}\rho,$$

in $L_{q^2,\frac{\mu}{2}-\alpha}(\mathbb{R}_{q^2,+})\cap L_{q^2,\frac{\nu}{2}-\beta}(\mathbb{R}_{q^2,+})\cap L_{q^2,\frac{\nu}{2}-\beta-\gamma}(\mathbb{R}_{q^2,+})$, *for* γ *satisfying*

$$1+\Re(\nu)>\Re(\gamma)>\ max\,\{0,\Re(\nu-\lambda)\}.$$

Now, we shall solve the system of triple q^2-integral equations (4.1)–(4.3). Since the function $g(\rho)$ is only defined in $A_{q^2}\cap B_{q^2}$, we can write

$$g(\xi)=g_1(\xi)+g_2(\xi),$$

g_1 and g_2 defined in A_{q^2} and B_{q^2} respectively. So, we may assume that

$$\psi=A_1+A_2,$$

and we solve the equations in the form

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}[A_1(\rho)+A_2(\rho)]J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=f(\xi),\quad \xi\in A_{q^2}, \tag{4.6}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_1(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=g_1(\xi),\quad \xi\in A_{q^2}, \tag{4.7}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_2(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=g_2(\xi),\quad \xi\in B_{q^2}, \tag{4.8}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}[A_1(\rho)+A_2(\rho)]J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=h(\xi),\quad \xi\in B_{q^2}, \tag{4.9}$$

We rewrite the equations as two pairs of dual q-integral equations, namely

$$\begin{cases}\xi^{-\alpha}\displaystyle\int_0^\infty \rho^{-\alpha}A_1(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=g_1(\xi),\quad \xi\in A_{q^2},\\[2ex] \xi^{-\beta}\displaystyle\int_0^\infty \rho^{-\beta}A_1(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=h(\xi)-f_2(\xi),\quad \xi\in B_{q^2}\end{cases} \tag{4.10}$$

$$\begin{cases}\xi^{-\alpha}\displaystyle\int_0^\infty \rho^{-\alpha}A_2(\rho)J_\mu(\sqrt{\rho\xi};q^2)d_{q^2}\rho=g_2(\xi),\quad \xi\in B_{q^2},\\[2ex] \xi^{-\gamma}\displaystyle\int_0^\infty \rho^{-\gamma}A_2(\rho)J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=f(\xi)-f_1(\xi),\quad \xi\in A_{q^2},\end{cases} \tag{4.11}$$

where

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}A_1(\rho)J_\kappa(\sqrt{\rho\xi};q^2)d_{q^2}\rho=f_1(\xi),\qquad \xi\in A_{q^2},$$
$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}A_2(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho=f_2(\xi),\qquad \xi\in B_{q^2},$$

Then we can solve the first and second pairs by Theorem **??**. For the first pairs

$$A_1(\xi)=(1-q^2)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda(\sqrt{\rho\xi};q^2)I_{q^2}^{\mu/2+\alpha,\lambda-\mu}g_1(\rho)\,d_{q^2}\rho$$

$$+(1-q^2)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda(\sqrt{\rho\xi};q^2)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda}[h(\rho)-f_2(\rho)]\,d_{q^2}\rho,$$

where, $\lambda := \frac{1}{2}(\mu+\nu) - (\alpha-\beta) > -1$. The solution of the second pair has the form

$$A_2(\xi) = (1-q^2)^{\lambda-\mu+2\gamma-2}\xi^{\lambda/2-\kappa/2+\gamma}\int_0^a J_\lambda(\sqrt{\rho\xi};q^2) I_{q^2}^{\kappa/2+\gamma,\lambda-\kappa}[f(\rho)-f_1(\rho)]\,d_{q^2}\rho$$

$$+(1-q^2)^{\lambda-\mu-2}\xi^{\lambda/2-\kappa/2+\gamma}\int_a^\infty J_\lambda(\sqrt{\rho\xi};q^2)\mathcal{K}_{q^2}^{\lambda/2-\mu/2-\alpha,\mu-\lambda} g_2(\rho) d_{q^2}\rho,$$

where, $\lambda := \frac{1}{2}(\mu+\kappa) - (\gamma-\alpha) > -1$.

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